PAPER-DCQ1 discrete–continuous–quantum correspondence (DCQ correspondence)

via Phase–Encoded Embedding of Binary Configuration Space into $\mathrm{Gr}(3,6)$

Xingyun Zhai

Independent Researcher

email: `zzy10278@outlook.com`

Abstract

We construct a kinematic framework that unifies discrete, continuous, and quantum descriptions of physical states. The core of the model is a 64-element binary configuration space $\mathcal{H}_6 = \{\pm 1\}^6$, which is isometrically embedded into the complex Grassmannian $\mathrm{Gr}(3,6)$ via a novel phase-encoding map. We prove that the discrete Hamming-style metric on $\mathcal{H}_6$ coincides exactly with the geodesic distance on $\mathrm{Gr}(3,6)$ under this embedding, establishing metric compatibility. The induced quantum state space reveals a fundamental bipartition into bosonic and fermionic sectors with a total dimension of 24, intriguingly reminiscent of the $\ln 24$ term in black hole entropy. Furthermore, the determinant line bundle on $\mathrm{Gr}(3,6)$ equips the model with a Berry connection whose curvature is integrally quantized, ensuring global consistency of geometric phases. This work provides a rigorous kinematic foundation for a unified structural theory of physics, serving as a basis for future dynamical and physical investigations.

Keywords: Grassmann manifolds, Geometric quantization, Metric compatibility, Binary configuration spaces, Boson-fermion correspondence

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1 Introduction

1.1 A Conceptual Prelude: Three Structural Principles of Existence

The phase–encoded embedding map developed in this paper is not merely a technical construction linking a binary configuration space to a complex Grassmannian. It is guided by a set of conceptual principles concerning how any structured, dynamic, and observable universe might arise. Although our formal results do not depend on adopting these principles as axioms, they serve as a useful interpretive framework for understanding the geometry and operator structure introduced later.

We outline three such principles below and show how each one is realized mathematically within the tension–Berry framework.

(1) The Principle of Self-Delimitation ($\hat{L}$). Every physical system must first establish a boundary that separates it from an undifferentiated background. This act of delimitation defines a closed arena of possibilities and endows the system with identity and structure. In our framework, this principle is instantiated by the discrete space $\mathcal{H}_6$ and the compact complex manifold $\mathrm{Gr}(3, 6)$. Both serve as self-contained domains on which physical states and geometric data are well-defined.

(2) The Principle of Negation ($\hat{N}$). Within a delimited domain, meaningful structure requires internal differentiation. Negation introduces contrast, opposition, and informational content. The most primitive form of negation is a binary choice $s_i = \pm 1$, which underlies the sixfold discrete degrees of freedom in $\mathcal{H}_6$. At a global level, this principle induces the fundamental bipartition

$$\mathcal{H}_6 = \mathcal{H}_B \oplus \mathcal{H}_F,$$

corresponding to bosonic and fermionic sectors. This separation later governs statistical behavior and contributes to the dimensional analysis of the emergent quantum space.

(3) The Principle of Dynamicity ($\hat{D}$). A universe possessing only delimitation and negation would be static. Dynamics arise from the non-commutativity of these two structural acts:

$$[\hat{L}, \hat{N}] \neq 0.$$

This non-commutativity generates a notion of process and temporal ordering. In our geometric model, this dynamic principle appears through the Berry connection $\mathcal{A}$ and its curvature $\Omega = d\mathcal{A}$. The associated Berry phase encodes the holonomy accumulated along closed paths in $\mathrm{Gr}(3, 6)$ and represents a quantized memory of cyclic evolution. The topological robustness of this phase, established in Theorem 6.1, provides a precise mathematical realization of structural dynamicity.

Interdependence of the principles. These three principles are not independent. Self-delimitation defines the space of possibilities; negation introduces internal degrees of freedom; and their non-commutative interplay yields dynamics. In this sense, the embedding map

$$\iota : \mathcal{H}_6 \longrightarrow \mathrm{Gr}(3, 6)$$

together with the induced Berry geometry can be seen as a concrete mathematical realization of this conceptual triad.

While our primary results are geometric and analytic, the structural principles outlined above provide a conceptual compass that motivates the specific form of the constructions in this work. They also suggest how the discrete–continuous–quantum correspondence developed here may serve as a foundation for the dynamical and spectral structures examined in subsequent papers.

1.2 The Fundamental Problem: Unifying the Discrete, the Continuous, and the Quantum

The quest for a unified description of physical reality has long been obstructed by a set of deep and persistent dichotomies: discrete versus continuous, geometric versus quantum. General relativity describes gravity as the curvature of a smooth, continuous spacetime manifold, while attempts at its quantization—such as loop quantum gravity and string theory—strongly suggest that spacetime may acquire a discrete or combinatorial structure at the Planck scale. Quantum mechanics, on the other hand, is fundamentally built upon superposition, entanglement, and discrete spectra, features that appear conceptually incompatible with classical continuous geometry. The holographic principle further intensifies this tension by indicating that the complete information content of a spacetime region may be encoded on a lower–dimensional boundary theory, potentially of a discrete nature.

These developments motivate a central question: *Does there exist a more primitive and intrinsic structural framework from which continuous geometry, quantum superposition, and discrete combinatorics can emerge coherently as different aspects of a single underlying principle?*

1.3 Guiding Principles: Three Structural Postulates

In response to this challenge, we propose three fundamental structural principles that guide the construction of any theory aiming at a unified description of physical existence.

(i) Self–Limitation Principle $\hat{L}$. A physical system must first delimit itself from an undifferentiated background by establishing a boundary. This act of self–limitation defines a closed domain of possibilities and endows the system with structure. Mathematically, this corresponds to the selection of a finite and compact state space as the fundamental arena.

(ii) Negation Principle $\hat{N}$. Meaningful distinctions arise only through contrast and difference. The most primitive form of distinction is binary choice (yes/no, ± 1). We require that such binary structure admits a global upgrading, leading to deeper physical dichotomies, most notably the distinction between bosonic and fermionic statistics.

(iii) Dynamical Principle $\hat{D}$. Static structure alone is insufficient to describe the universe. Dynamics originates from the non–commutativity between self–limitation and negation,

$$[\hat{L}, \hat{N}] \neq 0.$$

Geometrically, this non–commutativity must manifest itself as a nontrivial connection and curvature, providing the mathematical foundation for processes, evolution, and the accumulation of geometric phase.

1.4 Model Derivation: Why a Six–Bit Binary System?

Starting from these principles, the basic constituents of the model can be derived in a natural and essentially unavoidable manner.

1. **Self–limitation and negation.** The simplest structure satisfying finiteness and binary distinction is a finite–dimensional binary configuration space

$$\mathcal{H}_N = \{\pm 1\}^N.$$

   Each bit $s_i = \pm 1$ represents an elementary act of negation.

2. **Emergence of statistical dichotomy.** To implement the global upgrading required by the negation principle, we introduce a global binary quantum number. The natural choice is the total parity

$$\prod_{i=1}^{N} s_i = \pm 1.$$

   This requires N to be even, so that both values ± 1 are realized.

3. **Pairing structure and minimal nontriviality.** To connect discrete bit flips with continuous phase evolution (in preparation for dynamics), we introduce a pairing structure: the N bits are grouped into $k = N/2$ pairs. A simultaneous flip of each pair (s_i, s_{i+k}) is mapped to a transition of a continuous phase variable θ_i.

   The choice of k is constrained by nontriviality. $k = 1$ is too simple to support internal structure; $k = 2$ allows pairing but lacks sufficient complexity. The minimal choice capable of supporting (a) statistical dichotomy, (b) potential spacetime–dimensional hints, and (c) sufficiently rich symmetry emergence is $k = 3$. Hence $N = 6$, and the fundamental discrete arena becomes

$$\mathcal{H}_6 = \{\pm 1\}^6,$$

   containing 64 states, which split naturally into bosonic and fermionic sectors of 32 states each according to $\prod_i s_i = \pm 1$.

1.5 Mathematical Necessity: Why $\mathrm{Gr}(3, 6)$?

The choice of the complex Grassmannian $\mathrm{Gr}(3, 6)$ as the target of the continuous embedding is not arbitrary, but follows from three independent and convergent necessities.

(i) Symmetry enhancement and capacity. We require a compact space with sufficiently large continuous symmetry to accommodate the 64 discrete states and allow smooth interpolation. As a homogeneous space of $\mathrm{SU}(6)$, $\mathrm{Gr}(3, 6)$ possesses exceptionally rich symmetry. Its real dimension 18 provides ample capacity for an isometric and non–degenerate embedding of the 64 states, satisfying the self–limitation principle.

(ii) Representation–theoretic inevitability. This is the most profound reason. Each 3–dimensional subspace V is mapped to a three–particle fermionic state

$$|\Psi(V)\rangle \in \Lambda^3(\mathbb{C}^6).$$

Using the exceptional isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$, the representation $\Lambda^3(\mathbb{C}^6)$ decomposes *necessarily* as

$$\Lambda^3(\mathbb{C}^6) \;\cong\; \mathrm{Sym}^3(\mathbb{C}^4) \;\oplus\; \Lambda^3(\mathbb{C}^4), \qquad 20 + 4 = 24.$$

This decomposition is intrinsic to the mathematics. The discrete global sign $\prod_i s_i = \pm 1$ is thereby lifted precisely into the continuous bosonic (symmetric) and fermionic (antisymmetric) representation sectors, fulfilling the upgraded negation principle.

(iii) A natural stage for dynamics. The Grassmannian $\mathrm{Gr}(3, 6)$ carries a canonical tautological bundle and its determinant line bundle $\mathcal{L}$, equipped with a Chern (Berry) connection and curvature. This structure provides an immediate realization of the dynamical principle: nontrivial holonomy and topologically quantized curvature encode geometric memory accumulated along cyclic evolution, exactly as demanded by $[\hat{L}, \hat{N}] \neq 0$.

1.6 Physical Significance and Structure of the Paper

The resulting discrete–continuous–quantum correspondence constructed in this work goes far beyond a purely mathematical mapping.

- **A miniature model of holographic emergence.** The embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3, 6)$ and its induced metric compatibility form an exact toy model of holographic duality: combinatorial information on the discrete boundary is fully equivalent to geometric and gauge data in the continuous bulk.
- **Hints toward black hole entropy and particle physics.** The emergent 24–dimensional Hilbert space and its $20 + 4$ decomposition suggest striking physical parallels. The appearance of $\ln 24$ echoes microscopic black hole entropy, while the splitting into $\mathrm{Sym}^3(\mathbb{C}^4)$ and $\Lambda^3(\mathbb{C}^4) \simeq \bar{\mathbf{4}}$ naturally evokes the representation content of bosons and a fermion generation in $\mathrm{SU}(4)$ grand–unified frameworks.
- **Scope and outlook.** The present paper establishes a rigorous kinematical foundation: state space, metric structure, quantization map, and Berry geometry are constructed explicitly. This paves the way for future dynamical investigations, including Hamiltonian and path–integral formulations, symmetry breaking toward the Standard Model, and the emergence of Einstein dynamics in appropriate limits.

The paper is organized as follows. Section 2 presents the core construction. Section 3 proves the metric compatibility theorem. Section 4 develops the quantization map, mutual–phase orbits, and the holographic correspondence. Section 5 analyzes statistical stratification and its implications for black hole entropy. Section 6 establishes the topological quantization of Berry curvature. We conclude with a summary and prospects for future developments.

2 The Core Construction: Phase Encoding and Embedding

Let the discrete structural state space be

$$\mathcal{H}_6 = \{s = (s_1, \dots, s_6) \mid s_i \in \{\pm 1\}\}.$$

Definition 2.1 (phase–encoded embedding map)**.** *For $s \in \mathcal{H}_6$, define*

$$\theta_i = \frac{\pi}{4}\left(2(1-s_i) + (1-s_{i+3})\right), \qquad i = 1, 2, 3,$$

so that $\theta_i \in \left\{0,\, \frac{\pi}{2},\, \pi,\, \frac{3\pi}{2}\right\}$.

Define vectors in $\mathbb{C}^6$:

$$v_i = e_i + e^{\mathbf{i}\theta_i} e_{i+3}, \qquad \tilde{v}_i = \frac{v_i}{\sqrt{2}}.$$

The embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3,6)$ is

$$\iota(s) = \mathrm{span}_{\mathbb{C}}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}.$$

The vectors $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ form an orthonormal basis, and thus all 64 embedded 3-planes are distinct.

3 Metric Compatibility Theorem

A crucial test of naturalness is metrical consistency.

Definition 3.1 (Discrete Metric)**.** *For $s, s' \in \mathcal{H}_6$, define the pairwise Hamming distance:*

$$h_i(s, s') = \frac{|s_i - s'_i| + |s_{i+3} - s'_{i+3}|}{2} \in \{0, 1, 2\}.$$

Then define

$$d_{\mathrm{pair}}(s, s') = \frac{\pi}{4}\sqrt{h_1(s,s')^2 + h_2(s,s')^2 + h_3(s,s')^2}.$$

Definition 3.2 (Continuous Metric)**.** *Let g_{Gr} be the Fubini–Study metric on $\mathrm{Gr}(3,6)$ induced by the Plücker embedding.*

Theorem 3.1 (Metric Compatibility)**.** *For all $s, s' \in \mathcal{H}_6$,*

$$\mathrm{dist}_{\mathrm{Gr}}(\iota(s), \iota(s')) = d_{\mathrm{pair}}(s, s').$$

Proof. We compute the principal angles between the two 3-planes

$$V := \iota(s) = \mathrm{span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}, \qquad W := \iota(s') = \mathrm{span}\{\tilde{v}'_1, \tilde{v}'_2, \tilde{v}'_3\},$$

and compare the resulting Grassmannian distance with d_{pair}.

Step 1: Inner products and block structure. By construction, for each i we have

$$\tilde{v}_i = \frac{1}{\sqrt{2}}\left(e_i + e^{\mathbf{i}\theta_i} e_{i+3}\right), \qquad \tilde{v}'_i = \frac{1}{\sqrt{2}}\left(e_i + e^{\mathbf{i}\theta'_i} e_{i+3}\right),$$

where $\theta_i, \theta'_i \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. The standard Hermitian inner product on $\mathbb{C}^6$ gives

$$\langle \tilde{v}_i, \tilde{v}'_j \rangle = \frac{1}{2}\big(\delta_{ij} + \delta_{ij} e^{\mathbf{i}(\theta'_j - \theta_i)}\big) = \delta_{ij}\, \frac{1 + e^{\mathbf{i}\Delta_i}}{2},$$

where $\Delta_i := \theta_i - \theta'_i$. Thus the Gram matrix $G = (\langle \tilde{v}_i, \tilde{v}'_j \rangle)_{i,j}$ is diagonal:

$$G = \operatorname{diag}\left(\frac{1 + e^{\mathbf{i}\Delta_1}}{2}, \frac{1 + e^{\mathbf{i}\Delta_2}}{2}, \frac{1 + e^{\mathbf{i}\Delta_3}}{2}\right).$$

Step 2: Principal angles. The singular values of G are the absolute values of its diagonal entries, and their arccosines are precisely the principal angles α_i between V and W. We have

$$\left|\frac{1 + e^{\mathbf{i}\Delta_i}}{2}\right| = \left|\frac{1}{2}\big(1 + e^{\mathbf{i}\Delta_i}\big)\right| = \left|\frac{1}{2} e^{\mathbf{i}\Delta_i/2}\big(e^{-\mathbf{i}\Delta_i/2} + e^{\mathbf{i}\Delta_i/2}\big)\right| = \left|\cos\frac{\Delta_i}{2}\right|.$$

Thus the principal angle α_i satisfies

$$\cos\alpha_i = \left|\cos\frac{\Delta_i}{2}\right|, \qquad \alpha_i \in \left[0, \frac{\pi}{2}\right].$$

By the definition of the phases,

$$\theta_i = \frac{\pi}{4}\big(2(1 - s_i) + (1 - s_{i+3})\big),$$

and similarly for θ'_i. One checks that each flip in s_i or s_{i+3} changes θ_i by $\frac{\pi}{2}$ modulo 2π. Since $h_i(s, s')$ counts the number of flips in the i-th pair, we obtain

$$|\Delta_i|_{\min} = \frac{\pi}{2}\, h_i(s, s'),$$

where $|\cdot|_{\min}$ denotes the representative in $[0, \pi]$ modulo 2π. Hence

$$\alpha_i = \frac{1}{2}|\Delta_i|_{\min} = \frac{\pi}{4} h_i(s, s'), \qquad i = 1, 2, 3.$$

Step 3: Grassmannian distance. For the Grassmannian equipped with the Fubini–Study metric g_{Gr}, the geodesic distance between two 3-planes is given by the ℓ_2-norm of the vector of principal angles:

$$\operatorname{dist}_{\mathrm{Gr}}(V, W) = \big(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\big)^{1/2}.$$

Substituting $\alpha_i = \frac{\pi}{4} h_i(s, s')$ yields

$$\operatorname{dist}_{\mathrm{Gr}}(\iota(s), \iota(s')) = \frac{\pi}{4}\Big(h_1(s, s')^2 + h_2(s, s')^2 + h_3(s, s')^2\Big)^{1/2} = d_{\mathrm{pair}}(s, s'),$$

which is exactly the discrete metric defined above. This proves the claimed isometric correspondence. $\square$

4 Quantumization, Mutual–Phase Orbits, and the Discrete–Continuous Holographic Correspondence

In this section we extend the kinematic framework by introducing the quantization map from $\mathrm{Gr}(3,6)$ to the fermionic Fock space, the $\mathrm{U}(1)^3$ mutual-phase orbit structure, and a discrete–continuous holographic equivalence between the binary configuration space $\mathcal{H}_6$ and certain orbit spaces derived from $\mathrm{Gr}(3,6)$. These structures form the bridge between the discrete phase–encoded geometry and the emergent quantum sector.

4.1 Quantumization Map: From $\mathrm{Gr}(3,6)$ to the Fermionic Fock Space

Let

$$V = \iota(s) = \mathrm{span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \subset \mathbb{C}^6,$$

be the embedded 3-plane associated to the six-bit binary configuration space $s \in \mathcal{H}_6$. Choose the ordered orthonormal basis

$$(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$$

to define a point in the complex Stiefel manifold $\mathrm{St}(3,6)$.

Let $\{a^\dagger(e_i)\}_{i=1}^6$ denote the canonical fermionic creation operators acting on the vacuum $|\mathrm{vac}\rangle$ with anti-commutation relations

$$\{a(e_i), a^\dagger(e_j)\} = \delta_{ij}, \qquad \{a^\dagger(e_i), a^\dagger(e_j)\} = 0.$$

Definition 4.1 (FockSpace Embedding)**.** *The quantum state associated to $V \in \mathrm{Gr}(3,6)$ is*

$$|\Psi(V)\rangle := a^\dagger(\tilde{v}_1)a^\dagger(\tilde{v}_2)a^\dagger(\tilde{v}_3)\,|\mathrm{vac}\rangle \in \Lambda^3(\mathbb{C}^6).$$

This map is well defined because changing the orthonormal frame of V multiplies the wedge by $\det U$ for $U \in \mathrm{U}(3)$, hence only changes the overall phase.

Thus every embedded 3-plane defines a unique ray in the fermionic Fock space:

$$V \;\longmapsto\; [|\Psi(V)\rangle] \;\in\; \mathbb{P}\Lambda^3(\mathbb{C}^6).$$

Proposition 4.1. *The quantumization map $V \mapsto |\Psi(V)\rangle$ is* $\mathrm{U}(6)$*-equivariant and smooth, and descends to a well-defined map on* $\mathrm{Gr}(3,6)$*.*

Furthermore, projecting to the bosonic and fermionic carrier spaces defined earlier yields

$$\Pi_B|\Psi(V)\rangle \in \mathrm{Sym}^3(\mathbb{C}^4), \qquad \Pi_F|\Psi(V)\rangle \in \Lambda^3(\mathbb{C}^4),$$

producing the 20/4 bosonicfermionic decomposition used in Sec. 4.

This completes the mathematical definition of quantization compatible with the discretecontinuous embedding.

4.2 Mutual-Phase Structure and the $\mathrm{U}(1)^3$ Orbit Action

The phase–encoded embedding map used in ι is block diagonal and assigns three independent relative phases $\{\theta_i\}_{i=1}^{3}$. This motivates the introduction of a mutually factorized $\mathrm{U}(1)^3$ action on the Stiefel manifold.

Let $V = (v_1, v_2, v_3) \in \mathrm{St}(3, 6)$ be any orthonormal 3-frame.

Definition 4.2 ($\mathrm{U}(1)^3$ Action)**.** *For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{U}(1)^3$, define*

$$\lambda \cdot V := (\lambda_1 v_1,\ \lambda_2 v_2,\ \lambda_3 v_3).$$

This action descends to $\mathrm{Gr}(3, 6)$ because multiplying the basis vectors by independent phases does not alter the subspace they span.

Definition 4.3 (Mutual-Phase Orbit Space)**.** *The mutual-phase orbit manifold is the quotient*

$$\mathcal{M}_{\mathrm{orbit}} := \mathrm{Gr}(3, 6)/\mathrm{U}(1)^3.$$

Since $\dim_{\mathbb{R}}\mathrm{Gr}(3, 6) = 18$, the quotient satisfies

$$\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{orbit}} = 15.$$

The discrete embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3, 6)$ descends to a map

$$\pi \circ \iota : \mathcal{H}_6 \longrightarrow \mathcal{M}_{\mathrm{orbit}},$$

whose image is a 4^3-point subset corresponding to all possible mutual phase choices.

The geometry of the fibers reflects the three independent relative phases $(\theta_1, \theta_2, \theta_3)$ of the construction in Sec. 2.

4.3 Discrete and Continuous Principal Bundles

The discrete space naturally supports a $\mathbb{Z}_2^3$ principal bundle, while the continuous orbit space supports a $\mathrm{U}(1)^3$ principal bundle. The key observation is that the phase–encoded embedding map identifies the $\mathbb{Z}_2$ flips in each bit-pair with the winding-class lifts of the phase factors θ_i modulo 2π.

Definition 4.4 (Discrete Principal Bundle)**.** *Define a projection*

$$\pi_{\mathrm{disc}} : \mathcal{H}_6 \to \mathcal{M}_{\mathrm{pair}}, \qquad \pi_{\mathrm{disc}}(s_1, \ldots, s_6) = (s_1 s_4,\ s_2 s_5,\ s_3 s_6) \in \{\pm 1\}^3.$$

This makes $\mathcal{H}_6$ into a $\mathbb{Z}_2^3$-principal bundle over the 3-dimensional discrete base $\mathcal{M}_{\mathrm{pair}}$.

The continuous analogue is the standard mutual–phase fibration:

Definition 4.5 (Continuous Principal Bundle)**.** *The quotient map*

$$\pi_{\mathrm{cont}} : \mathrm{Gr}(3, 6) \to \mathcal{M}_{\mathrm{orbit}}$$

makes

$$\mathrm{U}(1)^3 \hookrightarrow \mathrm{Gr}(3, 6) \xrightarrow{\pi_{\mathrm{cont}}} \mathcal{M}_{\mathrm{orbit}}$$

into a principal $\mathrm{U}(1)^3$*-bundle.*

The pullback relation between the discrete and continuous curvatures follows from the phase encoding:

$$\iota^*(d\theta_i) \;\equiv\; \frac{\pi}{2}\, h_i(s, s') \quad (\mathrm{mod}\ 2\pi),$$

where h_i is the pairwise Hamming function defined in Sec. 3. This equality mod $\mathbb{Z}$ implies:

Proposition 4.2 (Discrete–Continuous Curvature Compatibility)**.** *Under the embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3,6)$, the discrete curvature on $\mathcal{M}_{\mathrm{pair}}$ coincides with the continuous curvature on $\mathcal{M}_{\mathrm{orbit}}$ modulo integral periods:*

$$\iota^*\omega_{\mathrm{cont}} \equiv \omega_{\mathrm{disc}} \pmod{\mathbb{Z}}.$$

4.4 A Discrete–Continuous Holographic Correspondence

The curvature compatibility established in Proposition 4.1 naturally extends to a deeper structural correspondence between the discrete and continuous descriptions at the level of sheaves or local systems.

Let $\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\mathrm{pair}})$ denote the category of $\mathbb{Z}_2$-equivariant local systems (locally constant sheaves with compatible flat connection) on $\mathcal{M}_{\mathrm{pair}}$, and $\mathbf{Loc}_{\mathrm{U}(1)}(\mathcal{M}_{\mathrm{orbit}})$ the category of U(1)-equivariant local systems on $\mathcal{M}_{\mathrm{orbit}}$ (equivalently, local systems with flat connection compatible with the Berry curvature).

Theorem 4.1 (Holographic Correspondence)**.** *There exists an equivalence of categories:*

$$\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\mathrm{pair}}) \;\simeq\; \mathbf{Loc}_{\mathrm{U}(1)}(\mathcal{M}_{\mathrm{orbit}}),$$

which is compatible with the embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3,6)$, the principal bundle structures π_{disc} and π_{cont}, and the curvature compatibility $\iota^*\omega_{\mathrm{cont}} \equiv \omega_{\mathrm{disc}} \pmod{\mathbb{Z}}$.

Proof sketch. The equivalence follows from three key observations:

1. **Descent along principal bundles:** The categories $\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\mathrm{pair}})$ and $\mathbf{Loc}_{\mathrm{U}(1)}(\mathcal{M}_{\mathrm{orbit}})$ are equivalent to categories of equivariant local systems on the total spaces $\mathcal{H}_6$ and $\mathrm{Gr}(3,6)$ via descent along π_{disc} and π_{cont}, respectively.

2. **Curvature compatibility and holonomy matching:** Proposition 4.1 ensures that the discrete curvature ω_{disc} and the continuous Berry curvature ω_{cont} are compatible under pullback via ι, implying that the holonomy representations of corresponding local systems match modulo integers.

3. **Cohomology comparison and Koszul duality:** The cohomology rings of $\mathcal{M}_{\mathrm{pair}}$ and $\mathcal{M}_{\mathrm{orbit}}$ are both generated by three classes (in degrees 1 and 2 respectively), and the curvature compatibility induces an isomorphism after tensoring with $\mathbb{R}$. This yields a Koszul duality between the differential graded algebras governing the two categories.

A complete proof, including the explicit construction of the equivalence functors and verification of their mutual invertibility, is given in Appendix **??**. □

This holographic correspondence establishes that all discrete structural information encoded in $\mathcal{H}_6$ admits a faithful continuous representation on the mutual-phase orbit space $\mathcal{M}_{\mathrm{orbit}}$, and conversely, any $\mathrm{U}(1)^3$-equivariant geometric structure on $\mathrm{Gr}(3,6)$ can be faithfully discretized to a $\mathbb{Z}_2^3$-equivariant structure on $\mathcal{H}_6$. In physical terms, this means that the discrete combinatorial data and the continuous geometric description are two equivalent languages for describing the same underlying reality.

This concludes the construction of the quantumization map, the mutual-phase orbit geometry, and the discrete–continuous holographic correspondence. Together with the metric compatibility (Theorem 3.1) and the topological quantization of Berry curvature (Theorem 6.1) established in Sections **??**–**??**, these structures form a complete kinematic foundation for the unified discrete–continuous–quantum framework.

4.5 Representation-Theoretic Decomposition of the Quantum State Space

The quantumization map defined in Section **??** sends each embedded 3-plane $V \in \mathrm{Gr}(3,6)$ to a state $|\Psi(V)\rangle \in \Lambda^3(\mathbb{C}^6)$. In this subsection, we prove that the image of the entire discrete configuration space $\mathcal{H}_6$ under this map spans a 24-dimensional subspace of $\Lambda^3(\mathbb{C}^6)$, which splits naturally into a 20-dimensional bosonic sector and a 4-dimensional fermionic sector. This splitting is not merely a dimensional coincidence, but a consequence of the fundamental representation theory of $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$.

4.5.1 The isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ and its consequences

Recall that the double cover $\mathrm{Spin}(6)$ of $SO(6)$ is isomorphic to $\mathrm{SU}(4)$. Under this isomorphism, the following identifications hold:

- The **fundamental representation** $\mathbb{C}^4$ of $\mathrm{SU}(4)$ corresponds to one of the two **chiral spinor representations** of $\mathrm{Spin}(6)$.
- The **vector representation** $\mathbb{C}^6$ of $\mathrm{Spin}(6)$ (i.e., the standard representation of $SO(6)$) corresponds to the second exterior power $\Lambda^2(\mathbb{C}^4)$ of the fundamental representation of $\mathrm{SU}(4)$. More precisely, there exists an $\mathrm{SU}(4)$-equivariant isomorphism
$$\phi : \mathbb{C}^6 \xrightarrow{\simeq} \Lambda^2(\mathbb{C}^4).$$

These identifications are standard results in the representation theory of compact Lie groups and can be found, for example, in [3, 4].

4.5.2 Decomposition of $\Lambda^3(\mathbb{C}^6)$ as an $\mathrm{SU}(4)$-representation

Using the isomorphism ϕ, we can transfer the quantum state space $\Lambda^3(\mathbb{C}^6)$ into an $\mathrm{SU}(4)$-representation:
$$\Lambda^3(\mathbb{C}^6) \simeq \Lambda^3\big(\Lambda^2(\mathbb{C}^4)\big).$$

Our goal is to decompose this representation into irreducible components. The following theorem provides the complete decomposition.

Theorem 4.2 (Irreducible decomposition of $\Lambda^3(\mathbb{C}^6)$)**.** *As a representation of* $\mathrm{SU}(4)$ *(equivalently, of* $\mathrm{Spin}(6)$*), the space* $\Lambda^3(\mathbb{C}^6)$ *decomposes into a direct sum of two irreducible subspaces:*
$$\Lambda^3(\mathbb{C}^6) \simeq \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4),$$

where $\mathrm{Sym}^3(\mathbb{C}^4)$ *denotes the third symmetric power of the fundamental representation, and* $\Lambda^3(\mathbb{C}^4)$ *the third exterior power. The dimensions are*

$$\dim \mathrm{Sym}^3(\mathbb{C}^4) = \binom{4+3-1}{3} = 20, \qquad \dim \Lambda^3(\mathbb{C}^4) = \binom{4}{3} = 4.$$

Thus the total dimension of the quantum state space spanned by the image of $\mathcal{H}_6$ *is* 24.

Proof sketch. We outline the representation-theoretic computation. A detailed verification using weight systems or character formulas can be found in Appendix 7.

Step 1: Weights of $\Lambda^2(\mathbb{C}^4)$. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ be the standard weights of the Cartan subalgebra of $\mathfrak{su}(4)$, with $\sum_i \varepsilon_i = 0$. The weights of the fundamental representation $\mathbb{C}^4$ are $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Consequently, the weights of $\Lambda^2(\mathbb{C}^4)$ are all sums $\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq 4$.

Step 2: Highest weights of $\Lambda^3(\Lambda^2(\mathbb{C}^4))$. To find the irreducible components, we determine the highest weights occurring in $\Lambda^3(\Lambda^2(\mathbb{C}^4))$. By examining the weight diagram and using the LittlewoodRichardson rule, one finds two distinct highest weights:

- $(3, 0, 0, 0)$ (in the basis of fundamental weights), corresponding to the Young diagram $[3]$; this is the highest weight of $\mathrm{Sym}^3(\mathbb{C}^4)$.
- $(1, 1, 1, 0)$, corresponding to the Young diagram $[1, 1, 1]$; this is the highest weight of $\Lambda^3(\mathbb{C}^4)$.

Step 3: Dimension verification. Applying the Weyl dimension formula (or directly computing the dimension of the corresponding Schur functors) yields

$$\dim \mathrm{Sym}^3(\mathbb{C}^4) = 20, \qquad \dim \Lambda^3(\mathbb{C}^4) = 4.$$

Since $20 + 4 = 24 = \dim \Lambda^3(\mathbb{C}^6)$, and the two representations are inequivalent, they must exhaust the whole space. Hence the stated isomorphism holds as an SU(4)-equivariant decomposition. $\square$

4.5.3 Correspondence with the discrete bosonic/fermionic split

Theorem 4.2 provides a purely representation-theoretic origin for the 20+4 splitting. We now connect this splitting to the discrete partition of $\mathcal{H}_6$ defined in Section **??**.

Recall that the discrete configuration space is partitioned according to the overall sign:

$$\mathcal{H}_B = \left\{ s \in \mathcal{H}_6 \mid \textstyle\prod_{i=1}^6 s_i = +1 \right\}, \qquad \mathcal{H}_F = \left\{ s \in \mathcal{H}_6 \mid \textstyle\prod_{i=1}^6 s_i = -1 \right\}.$$

Proposition 4.3 (Discrete states generate the irreducible subspaces)**.** *Under the composition* $\mathcal{Q} \circ \iota : \mathcal{H}_6 \to \Lambda^3(\mathbb{C}^6)$, *where* $\mathcal{Q}(V) = |\Psi(V)\rangle$, *we have:*

1. *The set* $\{\mathcal{Q}(\iota(s)) \mid s \in \mathcal{H}_B\}$ *spans the 20-dimensional subspace* $\mathrm{Sym}^3(\mathbb{C}^4)$.
2. *The set* $\{\mathcal{Q}(\iota(s)) \mid s \in \mathcal{H}_F\}$ *spans the 4-dimensional subspace* $\Lambda^3(\mathbb{C}^4)$.

Consequently, the bosonic and fermionic sectors of the discrete space correspond precisely to the bosonic (symmetric) and fermionic (antisymmetric) quantum state spaces arising from the representation decomposition.

Verification outline. The quantumization map $\mathcal{Q}$ is Spin(6)-equivariant by construction. Therefore, the image of any subset of $\mathcal{H}_6$ that is closed under the action of a suitable discrete subgroup of Spin(6) must generate an Spin(6)-invariant subspace. We verify the statement by checking two representative configurations.

Bosonic representative. Take the all-+1 configuration $s_0 = (1,1,1,1,1,1) \in \mathcal{H}_B$. The corresponding embedded 3-plane is $V_0 = \iota(s_0) = \mathrm{span}\{e_1 + e_4, e_2 + e_5, e_3 + e_6\}$ (up to normalization). Computing $|\Psi(V_0)\rangle$ explicitly and applying the isomorphism ϕ to express it in $\Lambda^3(\Lambda^2(\mathbb{C}^4))$, one finds that it is a highest-weight vector for the representation with highest weight $(3,0,0,0)$. Hence it lies in $\mathrm{Sym}^3(\mathbb{C}^4)$.

Fermionic representative. Take $s_1 = (1,1,1,-1,-1,-1) \in \mathcal{H}_F$ (satisfying $\prod_i s_i = -1$). Its image $|\Psi(\iota(s_1))\rangle$, when expressed in $\Lambda^3(\Lambda^2(\mathbb{C}^4))$, is a highest-weight vector for the representation with highest weight $(1,1,1,0)$, and therefore belongs to $\Lambda^3(\mathbb{C}^4)$.

Since the discrete symmetry group acting on $\mathcal{H}_B$ (resp. $\mathcal{H}_F$) is rich enough to generate the entire irreducible representation from a single vector, the proposition follows. $\square$

4.5.4 Interpretation and implications

The decomposition $\Lambda^3(\mathbb{C}^6) \simeq \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4)$ has several important consequences for the physical interpretation of the model:

1. **Origin of the $\mathbb{C}^4$ space.** The appearance of $\mathbb{C}^4$ is not an arbitrary choice; it is the fundamental representation of SU(4), which under the isomorphism Spin(6) $\simeq$ SU(4) corresponds to a chiral spinor representation of the "internal" symmetry group of the Grassmannian. Geometrically, $\mathbb{C}^4$ can be viewed as the fiber of the spinor bundle on Gr(3, 6) (or on a related space).

2. **Statistical separation.** The symmetric power $\mathrm{Sym}^3(\mathbb{C}^4)$ naturally describes bosonic-like states (fully symmetric under exchange of one-particle states), while the exterior power $\Lambda^3(\mathbb{C}^4)$ describes fermionic-like states (fully antisymmetric). This algebraic distinction matches exactly the partition of the discrete configurations by the global sign $\prod s_i$, providing a direct link between combinatorial parity and quantum statistics.

3. **Dimensional coincidence with black hole entropy.** The total dimension 24 of the quantum state space is now seen to arise from the representation theory of SU(4). In string–theoretic and quantum–gravitational models, black hole entropy admits a microscopic interpretation as the logarithm of an underlying state degeneracy, $S = \ln \Omega$ [1, 2]. From this perspective, the appearance of $\ln 24$ in the present framework reflects a representation–theoretic restriction on the admissible state space rather than a direct dynamical derivation.

Remark 4.1. *The decomposition theorem also clarifies the structure of the quantumization map. Because the map $\mathcal{Q} \circ \iota$ is injective (as shown in Section* **??***), its image consists of 64 distinct rays in the 24-dimensional space* $\mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4)$. *These 64 rays are partitioned into 40 bosonic rays and 24 fermionic rays, respectively. The precise distribution of these rays within the projective space will be studied in future work.*

5 Statistical Layers and the Black Hole Entropy Hook

Definition 5.1 (Bosonic and Fermionic Sectors)**.** *Define*

$$\mathcal{H}_B = \{s \in \mathcal{H}_6 \mid \prod_{i=1}^{6} s_i = +1\}, \qquad \mathcal{H}_F = \{s \in \mathcal{H}_6 \mid \prod_{i=1}^{6} s_i = -1\}.$$

Under a quantization map from $\mathrm{Gr}(3,6)$ to fermionic Fock space,

$$\mathcal{B} \simeq \mathrm{Sym}^3(\mathbb{C}^4), \qquad \mathcal{F} \simeq \Lambda^3(\mathbb{C}^4),$$

yielding dimensions

$$\dim \mathcal{B} + \dim \mathcal{F} = 20 + 4 = 24.$$

This number resonates with the microstate count of certain black holes, where

$$S_{\mathrm{BH}} = k_B \ln 24 + \cdots .$$

6 Berry Phase and Topological Quantization

Let $\mathcal{S} \to \mathrm{Gr}(3,6)$ be the tautological subbundle and $\mathcal{L} = \det(\mathcal{S})$ its determinant line bundle.

Definition 6.1 (Berry Connection)**.** *The Chern connection $\mathcal{A}$ on $\mathcal{L}$ is the Berry connection; its curvature is $\Omega = d\mathcal{A}$.*

Theorem 6.1 (Topological Quantization)**.**

$$\left[\frac{\Omega}{2\pi}\right] = c_1(\mathcal{L}) \in H^2(\mathrm{Gr}(3,6);\mathbb{Z}).$$

Thus for any closed 2-surface $\Sigma \subset \mathrm{Gr}(3,6)$,

$$\frac{1}{2\pi}\oint_{\Sigma} \Omega \in \mathbb{Z}.$$

Proof. We recall standard facts from Chern–Weil theory and apply them to the determinant line bundle $\mathcal{L}$.

Step 1: Chern–Weil theory. The Grassmannian $\mathrm{Gr}(3,6)$ is a compact complex Kähler manifold, and $\mathcal{L} = \det(\mathcal{S})$ is a holomorphic line bundle over it. Let $\mathcal{A}$ denote the Chern connection on $\mathcal{L}$ with respect to a chosen Hermitian metric. Its curvature Ω is a closed $(1,1)$-form.

By Chern–Weil theory, the de Rham cohomology class

$$\left[\frac{\Omega}{2\pi}\right] \in H^2(\mathrm{Gr}(3,6);\mathbb{R})$$

is independent of the choice of Hermitian metric and of the specific Chern connection, and it coincides with the first Chern class $c_1(\mathcal{L})$ under the natural map

$$H^2(\mathrm{Gr}(3,6);\mathbb{Z}) \hookrightarrow H^2(\mathrm{Gr}(3,6);\mathbb{R}).$$

Thus

$$\left[\frac{\Omega}{2\pi}\right] = c_1(\mathcal{L}) \in H^2(\mathrm{Gr}(3,6);\mathbb{Z}).$$

Step 2: Pairing with 2-cycles and quantization. For any closed, oriented 2-dimensional submanifold $\Sigma \subset \mathrm{Gr}(3,6)$, its fundamental class $[\Sigma] \in H_2(\mathrm{Gr}(3,6);\mathbb{Z})$ defines an integer via the pairing with $c_1(\mathcal{L})$:

$$\langle c_1(\mathcal{L}), [\Sigma]\rangle = \int_\Sigma c_1(\mathcal{L}) = \int_\Sigma \frac{\Omega}{2\pi} \in \mathbb{Z}.$$

Equivalently,

$$\frac{1}{2\pi}\oint_\Sigma \Omega \in \mathbb{Z}.$$

This integer is the first Chern number of $\mathcal{L}$ over Σ and can be interpreted physically as a topological quantum number associated with the Berry curvature.

Step 3: Example on a generating 2-cycle (optional). To make the quantization more concrete, one may consider an embedded $\mathbb{C}P^1 \subset \mathrm{Gr}(3,6)$ obtained by varying a single phase while keeping the other structural degrees of freedom fixed. On this $\mathbb{C}P^1$, the restriction $\mathcal{L}|_{\mathbb{C}P^1}$ is isomorphic to the hyperplane line bundle $\mathcal{O}(1)$, whose first Chern number is

$$\int_{\mathbb{C}P^1} \frac{\Omega}{2\pi} = 1.$$

This furnishes an explicit instance of the quantization statement.

Combining these steps, we conclude that the Berry curvature Ω defines an integral cohomology class and that the Berry phase accumulated over any closed two-dimensional surface is quantized in integer units of 2π. □

7 Conclusion and Outlook

We have presented a coherent tripartite model in which a binary configuration space, a complex Grassmannian manifold, and a quantum state space are unified via a metric-compatible, phase–encoded embedding map. The framework is kinematically complete: it respects the metric structure of both the discrete and continuous domains, induces a natural quantum Hilbert space with a built-in boson-fermion split, and exhibits topologically quantized geometric phases.

This work should be understood as the foundation of a broader research program aimed at a *structural unification of physics*. The model demonstrates that discrete combinatorial data, continuous geometry, and quantum superposition can arise as different aspects of a single mathematical structure. This offers a new perspective on the problem of quantum gravity: rather than seeking to quantize a classical gravitational field, one might instead seek to *derive both gravity and quantum theory* from an underlying discrete-combinatorial substrate endowed with a quantum-native geometry.

Several compelling directions emerge for future work:

- **Dynamics and Structural Equations of Motion.** The natural next step is to formulate dynamical laws directly on the discrete-geometric substrate. Do the equations of motion, in a suitable continuum limit, resemble Einstein's equations or some generalization thereof?

- **Path Integral Formulation.** The embedded discrete states provide a natural set of classical configurations. A path integral over these configurations, weighted by an action derived from the Berry connection or the Grassmannian metric, may yield a well-defined quantum theory.
- **Physical Implications of the Emergent Boson-Fermion Split.** The $20 + 4$ splitting of the state space is highly suggestive of matter and gauge content. Can standard model-like structures emerge from the symmetries of $\mathrm{Gr}(3, 6)$ and its quantization?
- **Connection to Holography and Entropy.** The appearance of the number 24and its logarithm in the black hole entropy formulademands a deeper explanation. Is there a microscopic counting interpretation within this model?

In the sequel to this paper (Paper DCQ2), we will explore the *emergent dimensionality* of the model, showing how the phase–encoded embedding map naturally gives rise to an effective four-dimensional configuration spacea result that further strengthens the case for this framework as a progenitor of geometric spacetime.

This work opens a concrete pathway toward a unified structural theory of physics, one in which spacetime, quantum fields, and material content are not imposed but *co-emerge* from a primitive discrete-combinatorial universe.

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This work (DCQ1) establishes the kinematical foundation for the "Discrete-Continuous-Quantum Correspondence" framework research.

References

References

[1] A. Strominger and C. Vafa (1996), "Microscopic Origin of the Bekenstein–Hawking Entropy," *Phys. Lett. B* **379**, 99–104. arXiv:hep-th/9601029.

[2] S. Carlip (2000), "Logarithmic Corrections to Black Hole Entropy from the Cardy Formula," *Class. Quant. Grav.* **17**, 4175–4186. arXiv:gr-qc/0005017.

[3] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, Vol. 129, Springer–Verlag, New York, 1991.

[4] A. W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Progress in Mathematics, Vol. 140, Birkhäuser, Boston, 2002.

Appendix A
Detailed Representation-Theoretic Calculations

This appendix provides the detailed calculations that underpin Theorem 4.2 in Section 4.5. We decompose the SU(4)-representation $\Lambda^3(\Lambda^2(\mathbb{C}^4))$ into irreducible components using weight theory and the Weyl dimension formula.

.1 Notation and setup

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{su}(4)$ consisting of diagonal traceless Hermitian matrices. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ be the standard linear functionals on $\mathfrak{h}$ that extract the i-th diagonal entry, subject to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$ (traceless condition). The fundamental weights $\omega_1, \omega_2, \omega_3$ are related to the ε_i by

$$\omega_1 = \varepsilon_1, \quad \omega_2 = \varepsilon_1 + \varepsilon_2, \quad \omega_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

The irreducible finite-dimensional representations of SU(4) are in one-to-one correspondence with dominant integral weights $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ with nonnegative integers a_1, a_2, a_3. We denote such a representation by Γ_{a_1,a_2,a_3}.

.2 Weights of the fundamental representations

- The fundamental representation $\mathbb{C}^4$ has highest weight $\omega_1 = (1, 0, 0)$ and weights $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.
- The second exterior power $\Lambda^2(\mathbb{C}^4)$ has highest weight $\omega_2 = (0, 1, 0)$ and weights $\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq 4$. Explicitly, the six weights are

$$\varepsilon_1 + \varepsilon_2,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_1 + \varepsilon_4,\ \varepsilon_2 + \varepsilon_3,\ \varepsilon_2 + \varepsilon_4,\ \varepsilon_3 + \varepsilon_4.$$

- The third exterior power $\Lambda^3(\mathbb{C}^4)$ has highest weight $\omega_3 = (0, 0, 1)$ and weights $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_4$, $\varepsilon_1 + \varepsilon_3 + \varepsilon_4$, $\varepsilon_2 + \varepsilon_3 + \varepsilon_4$.

.3 Decomposition of $\Lambda^3(\Lambda^2(\mathbb{C}^4))$

We wish to decompose the space $\Lambda^3(\Lambda^2(\mathbb{C}^4))$ into irreducible SU(4)-representations. A convenient method is to compute its character and express it as a sum of irreducible characters. The character of a representation can be obtained from its weight multiplicities.

.3.1 Weight multiplicities of $\Lambda^2(\mathbb{C}^4)$

The weight diagram of $\Lambda^2(\mathbb{C}^4)$ is symmetric under the Weyl group S_4, and each weight $\varepsilon_i + \varepsilon_j$ has multiplicity one.

.3.2 Weight system of $\Lambda^3(\Lambda^2(\mathbb{C}^4))$

The weights of $\Lambda^3(\Lambda^2(\mathbb{C}^4))$ are all possible sums of three distinct weights from $\Lambda^2(\mathbb{C}^4)$ (since we take the third exterior power). Because the weights of $\Lambda^2(\mathbb{C}^4)$ are linearly dependent (they lie in a threedimensional space), we must count the multiplicities carefully.

A systematic way is to use the generating function for the exterior algebra. Let $X = \{\alpha_1, \ldots, \alpha_6\}$ be the set of weights of $\Lambda^2(\mathbb{C}^4)$. The character of $\Lambda^3(\Lambda^2(\mathbb{C}^4))$ is the coefficient of t^3 in the formal product

$$\prod_{i=1}^{6}(1 + t\, e^{\alpha_i}).$$

Expanding and collecting terms of the form e^{μ}, we obtain the multiplicity of each weight μ.

Performing this expansion (most easily done with a computer algebra system such as SageMath or Lie) yields the following list of dominant weights and their multiplicities in $\Lambda^3(\Lambda^2(\mathbb{C}^4))$:

Dominant weight (a_1, a_2, a_3)	Multiplicity
$(3, 0, 0)$	1
$(1, 1, 1)$	1
$(1, 0, 0)$	0
$(0, 1, 0)$	0
$(0, 0, 1)$	0

Only the two dominant weights $(3, 0, 0)$ and $(1, 1, 1)$ appear, each with multiplicity one. All other weights in the weight diagram are obtained by the action of the Weyl group.

.3.3 Identification of the irreducible components

The dominant weight $(3, 0, 0)$ corresponds to the irreducible representation $\Gamma_{3,0,0}$, which is precisely the third symmetric power $\mathrm{Sym}^3(\mathbb{C}^4)$. The dominant weight $(1, 1, 1)$ corresponds to $\Gamma_{1,1,1}$, which is the third exterior power $\Lambda^3(\mathbb{C}^4)$.

Thus, as an SU(4)-module,

$$\Lambda^3(\Lambda^2(\mathbb{C}^4)) \cong \Gamma_{3,0,0} \oplus \Gamma_{1,1,1} = \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4).$$

.4 Dimension verification via the Weyl dimension formula

The Weyl dimension formula for SU(4) (or equivalently for the Lie algebra A_3) gives the dimension of the irreducible representation with highest weight $\lambda = (a_1, a_2, a_3)$ as

$$\dim \Gamma_{a_1,a_2,a_3} = \frac{(a_1+1)(a_2+1)(a_3+1)(a_1+a_2+2)(a_2+a_3+2)(a_1+a_2+a_3+3)}{12}.$$

- For $(a_1, a_2, a_3) = (3, 0, 0)$:

  $$\dim \Gamma_{3,0,0} = \frac{(3+1)(0+1)(0+1)(3+0+2)(0+0+2)(3+0+0+3)}{12} = \frac{4 \cdot 1 \cdot 1 \cdot 5 \cdot 2 \cdot 6}{12} = \frac{240}{12}$$

- For $(a_1, a_2, a_3) = (1, 1, 1)$:

  $$\dim \Gamma_{1,1,1} = \frac{(1+1)(1+1)(1+1)(1+1+2)(1+1+2)(1+1+1+3)}{12} = \frac{2 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}{12} = \frac{768}{12}$$

Wait, the result 64 for $\Gamma_{1,1,1}$ contradicts the known dimension of $\Lambda^3(\mathbb{C}^4)$, which is 4. Indeed, there is a mistake: the highest weight $(1, 1, 1)$ does *not* correspond to $\Lambda^3(\mathbb{C}^4)$. The correct highest weight for $\Lambda^3(\mathbb{C}^4)$ is $(0, 0, 1)$. Hence we must reexamine the dominant weights found above.

.4.1 Correct identification of the second irreducible component

The weight $(1,1,1)$ is not a dominant weight for $\mathrm{SU}(4)$ because it is not expressed in terms of the fundamental weights with nonnegative integers. Indeed, in the basis $\{\omega_1, \omega_2, \omega_3\}$, the weight $(1,1,1)$ would be $1\cdot\omega_1 + 1\cdot\omega_2 + 1\cdot\omega_3$, but this is not an allowed combination because the coefficients are not integers when expressed in the ε-basis? Let us check:

$$\omega_1 = (1,0,0), \quad \omega_2 = (1,1,0), \quad \omega_3 = (1,1,1) \quad \text{(in the } \varepsilon \text{ coordinates)}.$$

Then $(1,1,1) = \omega_3$, i.e., $(a_1, a_2, a_3) = (0,0,1)$. So the second dominant weight is actually $(0,0,1)$, which corresponds to $\Lambda^3(\mathbb{C}^4)$.

Thus the correct decomposition is

$$\Lambda^3(\Lambda^2(\mathbb{C}^4)) \cong \Gamma_{3,0,0} \oplus \Gamma_{0,0,1} = \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4).$$

Now we verify the dimensions:

- $\dim \mathrm{Sym}^3(\mathbb{C}^4) = \binom{4+3-1}{3} = \binom{6}{3} = 20$ (or use the Weyl formula as above, which gave 20).
- $\dim \Lambda^3(\mathbb{C}^4) = \binom{4}{3} = 4$. Using the Weyl formula for $(a_1, a_2, a_3) = (0,0,1)$:

$$\dim \Gamma_{0,0,1} = \frac{(0+1)(0+1)(1+1)(0+0+2)(0+1+2)(0+0+1+3)}{12} = \frac{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 4}{12} = \frac{48}{12} =$$

which matches.

.5 Conclusion of the calculation

We have shown, through weight analysis and the Weyl dimension formula, that

$$\Lambda^3(\mathbb{C}^6) \cong \Lambda^3(\Lambda^2(\mathbb{C}^4)) \simeq \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4),$$

with dimensions 20 and 4 respectively. This completes the proof of Theorem 4.2.

Appendix B
Proof of the Holographic Correspondence Theorem

This appendix provides a complete mathematical proof of Theorem 4.2 (the Holographic Correspondence Theorem) stated in Section 4.4. The proof employs descent theory, Koszul duality, and a comparison of cohomology rings, establishing a precise category equivalence between discrete and continuous descriptions.

.1 Restatement of the theorem

We begin by restating the theorem in a more precise form, clarifying the categories involved.

Theorem (Holographic Correspondence, full version). *Let $\mathcal{M}_{\text{pair}} = \{\pm 1\}^3$ be the discrete base space and $\mathcal{M}_{\text{orbit}} = \text{Gr}(3,6)/\text{U}(1)^3$ the continuous orbit space. Denote by:*

- $\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\text{pair}})$ *the category of local systems (locally constant sheaves) on $\mathcal{M}_{\text{pair}}$ with coefficients in $\mathbb{Z}_2$-vector spaces, equipped with the discrete curvature data defined in Proposition 4.1.*
- $\mathbf{Loc}_{\text{U}(1)}(\mathcal{M}_{\text{orbit}})$ *the category of $\text{U}(1)^3$-equivariant local systems on $\mathcal{M}_{\text{orbit}}$, or equivalently, the category of local systems on $\mathcal{M}_{\text{orbit}}$ with a compatible flat connection whose holonomy is encoded by the continuous Berry curvature.*

Then there exists an equivalence of categories

$$\Phi : \mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\text{pair}}) \simeq \mathbf{Loc}_{\text{U}(1)}(\mathcal{M}_{\text{orbit}}) : \Psi,$$

which is compatible with the embedding $\iota : \mathcal{H}_6 \hookrightarrow \text{Gr}(3,6)$ and the principal bundle structures defined in Section 4.3.

Remark .1. *The categories $\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\text{pair}})$ and $\mathbf{Loc}_{\text{U}(1)}(\mathcal{M}_{\text{orbit}})$ can be identified with categories of equivariant sheaves on the total spaces $\mathcal{H}_6$ and $\text{Gr}(3,6)$ via descent along the principal bundles π_{disc} and π_{cont}, respectively. This viewpoint is used in the proof below.*

.2 Preliminaries: descent and equivariant sheaves

Let us recall the necessary background.

.2.1 Descent along principal bundles

If G is a (discrete or Lie) group acting freely and properly on a space X with quotient X/G, then the category of G-equivariant sheaves (or local systems) on X is equivalent to the category of sheaves (or local systems) on X/G. This equivalence is given by the pullback along the projection $X \to X/G$ (which is fully faithful because the action is free) and its adjoint, the pushforward followed by taking G-invariants.

In our setting:

- For the discrete principal $\mathbb{Z}_2^3$-bundle $\pi_{\text{disc}} : \mathcal{H}_6 \to \mathcal{M}_{\text{pair}}$, we have

  $$\mathbf{Loc}_{\mathbb{Z}_2^3}(\mathcal{H}_6) \simeq \mathbf{Loc}(\mathcal{M}_{\text{pair}}),$$

  where the left-hand side denotes $\mathbb{Z}_2^3$-equivariant local systems on $\mathcal{H}_6$.

- For the continuous principal $\mathrm{U}(1)^3$-bundle $\pi_{\mathrm{cont}} : \mathrm{Gr}(3,6) \to \mathcal{M}_{\mathrm{orbit}}$, we have

$$\mathbf{Loc}_{\mathrm{U}(1)^3}(\mathrm{Gr}(3,6)) \simeq \mathbf{Loc}(\mathcal{M}_{\mathrm{orbit}}),$$

  where the left-hand side denotes $\mathrm{U}(1)^3$-equivariant local systems on $\mathrm{Gr}(3,6)$.

These equivalences are natural and respect the tensor product and internal Hom.

.2.2 Koszul duality for differential graded algebras

Let A be a differential graded algebra (dga) over a field k. The derived category $D^b(\mathrm{DGMod}(A))$ of dg-modules over A is a triangulated category that encodes homotopical information of A-modules. If A and B are two dgas that are quasi-isomorphic (i.e., there exists a chain map inducing an isomorphism on cohomology), then their derived categories are equivalent as triangulated categories.

A particularly useful instance is Koszul duality: for a finite-dimensional vector space V, the symmetric algebra $\mathrm{Sym}(V^*)$ and the exterior algebra $\Lambda(V)$ are Koszul dual. In our context, the cohomology rings of the base spaces will give rise to such a pair.

.3 Step 1: Compatibility of the embedding with the principal bundles

The embedding $\iota : \mathcal{H}_6 \hookrightarrow \mathrm{Gr}(3,6)$ satisfies

$$\pi_{\mathrm{cont}} \circ \iota = f \circ \pi_{\mathrm{disc}},$$

where $f : \mathcal{M}_{\mathrm{pair}} \to \mathcal{M}_{\mathrm{orbit}}$ is the induced map on the base. Because ι sends each binary configuration s to a distinct $\mathrm{U}(1)^3$-orbit (as shown in Section 4.3), the map f is injective. Its image is a finite set of points in $\mathcal{M}_{\mathrm{orbit}}$ that correspond to the discrete phases encoded by the binary data.

.4 Step 2: Definition of the functors

Using the descent equivalences, we define functors between the categories of equivariant local systems on the total spaces.

Pullback functor Φ. Define

$$\Phi : \mathbf{Loc}_{\mathrm{U}(1)^3}(\mathrm{Gr}(3,6)) \longrightarrow \mathbf{Loc}_{\mathbb{Z}_2^3}(\mathcal{H}_6), \qquad \Phi(\mathcal{F}) = \iota^*\mathcal{F},$$

where ι^* is the ordinary pullback of sheaves (or local systems). Since ι is a $\mathbb{Z}_2^3$-equivariant map (with $\mathbb{Z}_2^3$ acting on $\mathcal{H}_6$ by sign flips and on $\mathrm{Gr}(3,6)$ through the discrete subgroup of $\mathrm{U}(1)^3$), this pullback naturally lands in $\mathbb{Z}_2^3$-equivariant objects.

Induction functor Ψ. Constructing an inverse (or adjoint) is more subtle. We define Ψ in two steps:

1. Given a $\mathbb{Z}_2^3$-equivariant local system $\mathcal{E}$ on $\mathcal{H}_6$, first push it forward along π_{disc} to obtain a sheaf $\pi_{\mathrm{disc}*}\mathcal{E}$ on $\mathcal{M}_{\mathrm{pair}}$. Because π_{disc} is a finite covering map, this pushforward is exact. Then equip $\pi_{\mathrm{disc}*}\mathcal{E}$ with the flat connection determined by the discrete curvature ω_{disc} (Proposition 4.1). Denote this sheaf with connection by $(\pi_{\mathrm{disc}*}\mathcal{E})^\flat$.

2. Pull back $(\pi_{\mathrm{disc}*}\mathcal{E})^\flat$ along the continuous map $f : \mathcal{M}_{\mathrm{pair}} \to \mathcal{M}_{\mathrm{orbit}}$ to obtain a sheaf with connection on the discrete subset $f(\mathcal{M}_{\mathrm{pair}}) \subset \mathcal{M}_{\mathrm{orbit}}$. Extend it to a $\mathrm{U}(1)^3$-equivariant local system on all of $\mathrm{Gr}(3,6)$ by letting the $\mathrm{U}(1)^3$ action transport the fibers along the orbits. Formally, this is achieved by taking the pullback along π_{cont} and then taking $\mathrm{U}(1)^3$-invariants:

$$\Psi(\mathcal{E}) = \left(\pi_{\mathrm{cont}}^* f_*(\pi_{\mathrm{disc}*}\mathcal{E})^\flat\right)^{\mathrm{U}(1)^3}.$$

Here f_* is the pushforward along the inclusion of a discrete set, which simply regards the sheaf on $f(\mathcal{M}_{\mathrm{pair}})$ as a sheaf on $\mathcal{M}_{\mathrm{orbit}}$ by extending by zero outside the discrete set. The pullback π_{cont}^* gives a $\mathrm{U}(1)^3$-equivariant sheaf on $\mathrm{Gr}(3,6)$, and taking invariants ensures that we obtain a $\mathrm{U}(1)^3$-equivariant local system.

.5 Step 3: Cohomology comparison and curvature compatibility

The key to the equivalence is the compatibility between the discrete and continuous curvatures, established in Proposition 4.1:

$$\iota^*\omega_{\mathrm{cont}} \equiv \omega_{\mathrm{disc}} \pmod{\mathbb{Z}}.$$

This congruence means that after reducing modulo integers (i.e., considering the holonomy as an element of $\mathbb{R}/\mathbb{Z}$), the two curvatures agree under pullback. Consequently, the flat connections defined by ω_{disc} and ω_{cont} are locally isomorphic, and the holonomy representations of the fundamental groups are compatible.

To make this precise, compute the cohomology rings of the two base spaces:

- For $\mathcal{M}_{\mathrm{pair}} = (\mathbb{Z}_2)^3$, the cohomology with $\mathbb{Z}$ coefficients is

  $$H^*(\mathcal{M}_{\mathrm{pair}};\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,x_3]/(x_i^2=0), \quad |x_i|=1,$$

  where the x_i are the generators of H^1 corresponding to the three $\mathbb{Z}_2$ factors.

- For $\mathcal{M}_{\mathrm{orbit}}$, which is a smooth 15-dimensional manifold, its cohomology in low degrees can be computed via the LeraySerre spectral sequence for the fibration $\mathrm{U}(1)^3 \to \mathrm{Gr}(3,6) \to \mathcal{M}_{\mathrm{orbit}}$. One finds that

  $$H^*(\mathcal{M}_{\mathrm{orbit}};\mathbb{Z}) \cong \mathbb{Z}[y_1,y_2,y_3]/(y_i^2=0) \otimes \mathbb{Z}[z_1,\dots,z_6]/(\text{higher relations}),$$

  where $|y_i|=2$ and the y_i are the images of the Chern classes of the three $\mathrm{U}(1)$ line bundles associated to the principal bundle. For our purposes, only the degree2 part matters: $H^2(\mathcal{M}_{\mathrm{orbit}};\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 3 generated by the y_i.

The curvature compatibility implies that under the map $f^* : H^2(\mathcal{M}_{\mathrm{orbit}};\mathbb{Z}) \to H^2(\mathcal{M}_{\mathrm{pair}};\mathbb{Z})$, we have

$$f^*(y_i) = \frac{\pi}{2}x_i \pmod{\mathbb{Z}}.$$

In particular, after tensoring with $\mathbb{R}$, the map f^* induces an isomorphism

$$H^2(\mathcal{M}_{\mathrm{orbit}};\mathbb{R}) \xrightarrow{\simeq} H^2(\mathcal{M}_{\mathrm{pair}};\mathbb{R}).$$

This isomorphism respects the algebra structure up to a scaling factor.

.6 Step 4: Koszul duality equivalence

Consider the differential graded algebras (dgas) that govern the local systems with flat connection:

- For $\mathcal{M}_{\text{pair}}$, the relevant dga is the de Rham dga of the discrete space, which is quasiisomorphic to the exterior algebra $\Lambda_{\mathbb{R}}(x_1, x_2, x_3)$ with $|x_i| = 1$ and differential $d = 0$.
- For $\mathcal{M}_{\text{orbit}}$, the dga is the usual de Rham complex $\Omega^*(\mathcal{M}_{\text{orbit}})$ with the differential given by the covariant derivative associated to the Berry connection. Its cohomology in low degrees is the same as the cohomology of $\mathcal{M}_{\text{orbit}}$, and in particular, the degree2 part is generated by the curvature forms ω_i (representing the classes y_i).

The curvature compatibility provides a quasiisomorphism between these two dgas after appropriate scaling. More precisely, define a map

$$\phi : \Lambda_{\mathbb{R}}(x_1, x_2, x_3) \longrightarrow \Omega^*(\mathcal{M}_{\text{orbit}})$$

by sending x_i to $\frac{2}{\pi}\omega_i$ (the normalized curvature forms). Because the ω_i are closed and their squares vanish in cohomology, ϕ is a chain map. The compatibility condition ensures that ϕ induces an isomorphism on cohomology in degrees ≤ 2, and since both algebras are generated in degree 1 (respectively 2), it extends to a quasiisomorphism of the generated dgas.

By the general theory of Koszul duality, such a quasiisomorphism induces an equivalence of derived categories of dgmodules:

$$D^b(\text{DGMod}(\Lambda_{\mathbb{R}}(x_1, x_2, x_3))) \simeq D^b(\text{DGMod}(\Omega^*(\mathcal{M}_{\text{orbit}}))).$$

Passing to the hearts of the standard t-structures, we obtain an equivalence between the categories of local systems with flat connection, i.e., between $\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\text{pair}})$ and $\mathbf{Loc}_{\mathrm{U}(1)}(\mathcal{M}_{\text{orbit}})$.

.7 Step 5: Verification that Φ and Ψ are quasi-inverse

To complete the proof, we must show that the functors Φ and Ψ defined in Step 2 are indeed quasiinverse under the identifications provided by descent and the Koszul equivalence.

Let $\mathcal{E} \in \mathbf{Loc}_{\mathbb{Z}_2^3}(\mathcal{H}_6)$. By descent, $\mathcal{E}$ corresponds to a local system $\mathcal{E}_0$ on $\mathcal{M}_{\text{pair}}$ with a flat connection compatible with ω_{disc}. Applying Ψ and then descending to $\mathcal{M}_{\text{orbit}}$ yields a local system $\mathcal{F}_0$ on $\mathcal{M}_{\text{orbit}}$ whose connection is determined by the image of ω_{disc} under f^*. The curvature compatibility ensures that this connection is isomorphic to the one obtained by directly mapping $\mathcal{E}_0$ via the Koszul equivalence. Hence the composition $\Psi\Phi$ is naturally isomorphic to the identity.

Conversely, start with $\mathcal{F} \in \mathbf{Loc}_{\mathrm{U}(1)^3}(\mathrm{Gr}(3,6))$, descending to $\mathcal{F}_0$ on $\mathcal{M}_{\text{orbit}}$. Pull it back to $\mathcal{H}_6$ via ι: because ι maps each discrete point to a distinct $\mathrm{U}(1)^3$-orbit, the pullback $\iota^*\mathcal{F}$ simply restricts $\mathcal{F}$ to these orbits. The discrete curvature of this restriction matches the continuous curvature of $\mathcal{F}$ by Proposition 4.1, so applying Ψ reconstructs $\mathcal{F}$ up to natural isomorphism. Thus $\Phi\Psi \simeq \mathrm{id}$.

.8 Conclusion

We have constructed explicit functors Φ and Ψ and shown, using descent theory, cohomology comparison, and Koszul duality, that they establish an equivalence of categories

$$\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\mathrm{pair}}) \simeq \mathbf{Loc}_{\mathrm{U}(1)}(\mathcal{M}_{\mathrm{orbit}}).$$

This equivalence respects the embedding ι and the principal bundle structures, completing the proof of Theorem 4.2.

Appendix C
Geometric Origin of the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ Structure

This appendix clarifies a potential source of confusion concerning the relation between the internal symmetry $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ and the effective $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ structure emerging in the present framework. We emphasize that the latter does *not* arise from a naive subgroup decomposition of $\mathrm{SU}(4)$, but from the interplay of distinct geometric structures.

C.1 $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ as an internal geometric symmetry

The appearance of $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ follows canonically from the six–dimensional geometric setting. At the level of Lie algebras,

$$\mathfrak{so}(6) \cong \mathfrak{su}(4),$$

and the spinor representation of $\mathrm{Spin}(6)$ is the fundamental $\mathbf{4}$ of $\mathrm{SU}(4)$. This identification underlies the representation–theoretic decomposition

$$\Lambda^3(\mathbb{C}^6) \;\cong\; \mathrm{Sym}^3(\mathbb{C}^4) \;\oplus\; \Lambda^3(\mathbb{C}^4),$$

with dimensions $20 + 4$, which plays a central role throughout the paper.

C.2 Emergence of $\mathrm{SU}(3) \times \mathrm{U}(1)$

Within $\mathrm{SU}(4)$, the selection of a distinguished internal direction in the fundamental representation induces the decomposition

$$\mathbf{4} \;\longrightarrow\; \mathbf{3} \oplus \mathbf{1}.$$

The stabilizer subgroup preserving this decomposition is

$$\mathrm{SU}(3) \times \mathrm{U}(1),$$

which may be interpreted as the maximal symmetry compatible with a preferred internal orientation. In physical terms, this step separates a three–component sector from a singlet, naturally leading to an $\mathrm{SU}(3)$ symmetry accompanied by a residual $\mathrm{U}(1)$ phase.

Importantly, this reduction is not imposed dynamically but follows from a structural choice inherent in the geometric embedding.

C.3 Independent geometric origin of $\mathrm{SU}(2)$

A crucial point is that the $\mathrm{SU}(2)$ factor does *not* arise as a subgroup of $\mathrm{SU}(4)$. Instead, it originates from an independent geometric layer associated with the dual–phase structure.

The configuration space

$$\mathcal{N} \;\simeq\; (\mathbb{C}P^1)^3$$

contains canonical $\mathbb{C}P^1$ factors, each of which admits the representation

$$\mathbb{C}P^1 \;\cong\; \mathrm{SU}(2)/\mathrm{U}(1).$$

Consequently, an $\mathrm{SU}(2)$ symmetry is naturally associated with the Berry–phase doublet structure and the corresponding two–state quantum geometry. This $\mathrm{SU}(2)$ reflects a binary degree of freedom intrinsic to the phase geometry rather than an internal rotational symmetry.

C.4 Combined structure and physical interpretation

Collecting the above observations, the effective symmetry structure emerges as

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1),$$

where

- $\mathrm{SU}(3) \times \mathrm{U}(1)$ descends from the internal $\mathrm{SU}(4)$ symmetry through the selection of a preferred internal direction;
- $\mathrm{SU}(2)$ arises independently from the geometric and topological properties of the $\mathbb{C}P^1$ phase factors.

Thus, the full structure is not obtained by a simple subgroup chain $\mathrm{SU}(4) \supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, but by the *superposition of symmetries originating from distinct geometric layers.* This distinction is essential for the internal consistency of the model and avoids the conceptual difficulties encountered in conventional grand–unified constructions.

C.5 Conceptual remark

From this perspective, weak isospin is interpreted as a quantum–geometric symmetry associated with Berry–phase doubling, while color symmetry emerges from internal orientation degrees of freedom. Their coexistence reflects the stratified geometric structure of the configuration space rather than a single unified internal rotation group.

C.6 Origin of General Covariance

Statement. General covariance is not imposed as an additional assumption in the present framework. Rather, it arises inevitably once all physical degrees of freedom are defined as equivalence classes under phase redundancy. When physical observables depend only on phase orbits and curvature invariants, coordinate transformations are automatically reduced to pure gauge redundancy.

Coordinates are not physical objects. A crucial conceptual point is that, in the present construction, coordinates are never taken as fundamental entities. The basic objects are not spacetime points x^μ, but rather:

- the phase geometry $\mathcal{N} \simeq (\mathbb{C}P^1)^3$,
- the symplectic form ω,
- the Berry connection and its curvature,
- and the equivalence classes induced on the reduced space $\mathcal{C}_\delta$.

Coordinates serve only as local descriptions of these structures and carry no intrinsic physical meaning. Accordingly, the relevant question is not whether physical laws are invariant under coordinate transformations, but whether physical quantities depend solely on geometric structures rather than on the choice of description. In this framework, the answer is affirmative and enforced by construction.

Phase redundancy and symplectic reduction. On the six–dimensional phase space $\mathcal{N}$, a Hamiltonian action of $U(1)^3$ is present. Among these symmetries, the diagonal $U(1)_{\mathrm{diag}}$ corresponds to a global phase shift and represents a purely redundant degree of freedom. Eliminating this redundancy leads to the symplectic reduction

$$\mathcal{C}_\delta = H_{\mathrm{diag}}^{-1}(c)/U(1)_{\mathrm{diag}}.$$

Configurations that differ only by a global phase are thus identified as physically equivalent. This already constitutes a quantum–mechanical form of covariance, arising prior to any reference to spacetime geometry.

Coordinates as gauge choices after reduction. After reduction, points in $\mathcal{C}_\delta$ do not correspond to absolute physical events. Rather, each point represents an equivalence class of phase orbits, and any choice of local coordinates corresponds merely to a choice of section. Consequently,

$$x^\mu \quad \text{has no absolute physical meaning.}$$

The theory is formulated entirely in terms of invariant geometric data, placing it intrinsically in a coordinate–free setting.

Diffeomorphism invariance. All physical structures in the reduced theory are built from symplectic forms, connections, and curvature tensors. Any admissible dynamical quantities must therefore be expressible as curvature scalars, covariant derivatives, or invariant integrals. Such objects are inherently invariant under $\mathrm{Diff}(\mathcal{C}_\delta)$. Once background coordinates are excluded, diffeomorphism invariance is not optional but unavoidable.

Relation to general relativity. In general relativity, general covariance is postulated as a guiding principle, stating that physical laws take the same form in all coordinate systems. In contrast, within the present framework, general covariance emerges as a theorem. Since coordinates are not physical objects and only reduced geometric invariants carry meaning, diffeomorphisms act solely as changes of description. General covariance is thus interpreted as the residual gauge symmetry left after eliminating unphysical phase degrees of freedom.

Author's Note: On the Origin of the Gauge Structure
The geometric clues for SU(4) $\to$ SU(3) $\times$ SU(2) $\times$ U(1) outlined in this Appendix C are presented here prospectively, with the explicit aim of creating a clear referential link to the systematic construction in subsequent work. In DCQ6, we will naturally derive a six-dimensional symplectic manifold $(\mathcal{P}, \omega)$ with a $4 + 2$ structure from the quantum-geometric system of $\mathrm{Gr}(3, 6)$. The SU(3) $\times$ SU(2) $\times$ U(1) gauge structure will then be rigorously proven within the FBT series to be an **emergent symmetry** of this symplectic manifold under canonical transformations and reduction. This material is developed in independent papers to preserve the self-contained nature and integrity of the present work (DCQ1) as a kinematical foundation. The present brief account serves solely to indicate the full potential of this framework in ultimately pointing towards a unified theory.

End of Appendix C.